POLYNOMIAL GROWTH OF BOHNENBLUST–HILLE CONSTANTS ON THE HAMMING CUBE

Abstract. We prove that the degree-m Bohnenblust–Hille constant C_m on the Hamming cube $\{\pm 1\}^n$ satisfies $C_m \le Km^{28}$ for an absolute $K < \infty$ and every n. The argument adapts the fixed-ratio / weighted bootstrap of Pellegrino–Teixeira (arXiv:2608.16584).

1. The inequality

Let $\{\pm 1\}^n$ carry the uniform product measure. Walsh characters are $\chi_S(x) = \prod_{i\in S} x_i$. A function $f : \{\pm 1\}^n \to \mathbb{C}$ of Walsh degree at most m expands as $f = \sum_{|S|\le m} \widehat{f}(S)\,\chi_S$. Write

$$q_r := \frac{2r}{r+1}, \qquad \frac{1}{q_r} = \frac{1}{2} + \frac{1}{2r},$$

and let C_r be the least constant, independent of n, such that every Walsh-homogeneous g of degree r satisfies

$$\|a(g)\|_{q_r} := \Big(\sum_{|S|=r} |\widehat{g}(S)|^{q_r}\Big)^{1/q_r} \le C_r\,\|g\|_\infty.$$

Finiteness of C_r is due to Defant–Mastyło–Pérez and to Eskenazis–Ivanisvili; the previously best growth was $C_m \le \exp\big(O(\sqrt{m\log m})\big)$. For $r < 12$ we use only that C_r is a fixed finite number (in particular $C_1 = 1$).

Theorem 1. *There is an absolute constant $K < \infty$ such that $C_m \le Km^{27}$ for every $m \ge 1$.*

Let $\widetilde{C}_m$ be the least n-independent constant such that every F of Walsh degree at most m satisfies

$$\Big(\sum_{|S|\le m} |\widehat{F}(S)|^{q_m}\Big)^{1/q_m} \le \widetilde{C}_m\,\|F\|_\infty.$$

Theorem 2. *There is an absolute constant $K < \infty$ such that $\widetilde{C}_m \le Km^{28}$ for every $m \ge 1$. Moreover, every such F satisfies the weighted bound*

$$\Big[\sum_{r=1}^{m} \frac{\|a(F^{=r})\|_{q_r}^2}{r^{10}}\Big]^{1/2} \le Km^{22}\,\|F\|_\infty. \tag{1}$$

The proof follows Pellegrino–Teixeira [1] section by section. Four holomorphic ingredients are replaced or dropped: Weissler by Bonami–Beckner; the torus projection [1, (4.5)] by a Markov bound on slices, used only for degrees $r < 12$; the split substitution $z_j \mapsto (x_j + y_j)/2$ by a fair 2-coloring (every cube monomial is already square-free); and dominant-power compression [1, Lemma 5.5], which is vacuous on the cube.

2. Preliminaries

Lemma 3 (Barycentric identity)**.** *If $m = d + e$ with $d, e \geq 1$, then*

$$\frac{1}{q_m} = \frac{d}{m}\frac{1}{q_d} + \frac{e}{m}\frac{1}{2} = \frac{d}{m}\frac{1}{2} + \frac{e}{m}\frac{1}{q_e}.$$

Lemma 4 (Mixed-norm interpolation [1, Lemmas 3.3–3.4])**.** *Let $a = (a_{AB})$ be a finite scalar array and set*

$$X_d(a) := \Big[\sum_A \Big(\sum_B |a_{AB}|^2\Big)^{q_d/2}\Big]^{1/q_d},$$

$$Y_e(a) := \Big[\sum_B \Big(\sum_A |a_{AB}|^2\Big)^{q_e/2}\Big]^{1/q_e}.$$

If $m = d + e$, then $\|a\|_{q_m} \leq X_d(a)^{d/m} Y_e(a)^{e/m}$.

The proof is pure ℓ_p interpolation and Minkowski ($q_e \leq 2$); it does not use holomorphy.

Lemma 5 (Vertex maximum)**.** *Let f be a $\mathbb{C}$-valued multilinear polynomial on $\mathbb{R}^n$. Then $\|f\|_{L^\infty([-1,1]^n)} = \|f\|_{L^\infty(\{\pm 1\}^n)}$.*

Proof. In each coordinate, $f = A + Bt$ with A, B independent of that coordinate. The map $t \mapsto |A + Bt|$ is convex on $\mathbb{R}$, hence maximized on $[-1, 1]$ at an endpoint. Iterating over coordinates yields a vertex. $\square$

Lemma 6 (Bonami–Beckner comparison)**.** *Let $k \geq 2$, $e \geq 1$, and let g be Walsh-homogeneous of degree e on $\{\pm 1\}^N$. Then*

$$\|g\|_2 \leq \Big(\frac{k+1}{k-1}\Big)^{e/2} \|g\|_{q_k}.$$

Proof. Bonami–Beckner with source $q_k \in (1, 2]$ and target 2 gives $\|T_\rho g\|_2 \leq \|g\|_{q_k}$ for $\rho = \sqrt{q_k - 1} = \sqrt{(k-1)/(k+1)}$. Homogeneity yields $T_\rho g = \rho^e g$. The endpoint $k = 1$ is $\rho = 0$ and is never used. $\square$

3. Two-block estimate

Lemma 7 (Cube two-block)**.** *Let $d, e \geq 2$, $m = d + e$, and let Q be Walsh-bihomogeneous of bidegree (d, e) on $\{\pm 1\}^{n_x} \times \{\pm 1\}^{n_y}$. Then*

$$\|a(Q)\|_{q_m} \leq 3\, C_d^{d/m} C_e^{e/m} \|Q\|_\infty.$$

Proof. For each $|A| = d$ set $g_A(y) = \sum_{|B|=e} \widehat{Q}(A, B)\, \chi_B(y)$. Parseval and Lemma 6 at $k = d$ give

$$\Big(\sum_B |\widehat{Q}(A, B)|^2\Big)^{1/2} \leq \Big(\frac{d+1}{d-1}\Big)^{e/2} \|g_A\|_{q_d}.$$

Raise to the power q_d, sum in A, and apply Tonelli:

$$X_d(a)^{q_d} \le \Big(\frac{d+1}{d-1}\Big)^{eq_d/2} \mathbb{E}_y \sum_A |g_A(y)|^{q_d}.$$

For each fixed vertex y, $x \mapsto Q(x,y)$ is homogeneous of degree d, so

$$\Big(\sum_A |g_A(y)|^{q_d}\Big)^{1/q_d} \le C_d \|Q(\,\cdot\,,y)\|_\infty \le C_d \|Q\|_\infty,$$

the second step a maximum over remaining vertices. Thus

$$X_d(a) \le \Big(\frac{d+1}{d-1}\Big)^{e/2} C_d \|Q\|_\infty,$$

and likewise for $Y_e(a)$. Lemma 4 yields the claimed geometric mean of C_d and C_e, multiplied by

$$\Big(\frac{d+1}{d-1}\Big)^{de/(2m)} \Big(\frac{e+1}{e-1}\Big)^{de/(2m)}.$$

The logarithm of this product is

$$\frac{de}{2m}\Big(\log\frac{d+1}{d-1} + \log\frac{e+1}{e-1}\Big).$$

On integers $d, e \ge 2$ the maximum is $\log 3$, attained at $(d,e) = (2,2)$ and in the limit $(2,\infty)$. Hence the prefactor is at most 3. No n appears, and no bidegree was extracted from a larger polynomial. $\square$

4. Coloring and capture

Color $[n] = X \sqcup Y$ independently and fairly. For $f = \sum_{|S|=r} \widehat{f}(S)\chi_S$ write $Q(x,y) = f(z(x,y))$. Each parent χ_S produces the unique child $\chi_{S\cap X}(x)\chi_{S\cap Y}(y)$ with the same coefficient (exact ancestry). Distinct parents never collide, and $\|Q\|_\infty = \|f\|_\infty$ identically.

Lemma 8 (Capture)**.** *For $0 < a < 1/2$ and $|S| = r$,*

$$p_r(a) := \mathbb{P}\big(ar \le |S\cap X| \le (1-a)r\big) \ge 1 - 2\exp\big(-2(\tfrac{1}{2}-a)^2 r\big).$$

Consequently

$$\mathbb{E}_\omega \sum_{(d,e)\in\mathcal{C}_r(a)} \|a(Q_{d,e})\|_{q_r}^{q_r} \ge p_r(a)\, \|a(f)\|_{q_r}^{q_r},$$

where $\mathcal{C}_r(a) = \{(d,e)\in\mathbb{N}^2 : d+e=r,\ ar \le d, e \le (1-a)r\}$.

Proof. The x-degree of χ_S is $\mathrm{Bin}(r,1/2)$. Hoeffding gives the displayed tail. The coefficient identity is Tonelli plus exact ancestry. The paper's $m^{-3/2}$ split and the central half-mass argument are unused: every cube monomial has $\max_j \alpha_j = 1$. $\square$

5. Weighted constant on slices

Definition 9. A *degree-m slice* of a Walsh-homogeneous parent f of degree m is any function $g(x) = f(x, a)$ obtained by freezing a subset of coordinates at a vertex a. Write $\mathcal{S}_m$ for this class. It is closed under further freezing. Lemma 5 gives $\|g\|_\infty \le \|f\|_\infty$.

For $B > 0$ and a parent f of degree m, let $\Gamma(f; B)$ be the least constant such that every $g \in \mathcal{S}_m$ of this parent satisfies

$$\Big[\sum_{r=1}^{m} \frac{\|a(g^{=r})\|_{q_r}^2}{r^{2B}}\Big]^{1/2} \le \Gamma(f; B)\, \|f\|_\infty. \tag{2}$$

The left-hand side is finite for each f (finitely many coefficients). No a-priori bound on a universal Γ_m is required.

Write $X_{d,e}$ and $Y_{d,e}$ for the mixed norms of a colored polynomial Q, and set

$$\widehat{X}_{d,e} := \Big(\frac{d-1}{d+1}\Big)^{e/2} X_{d,e}, \qquad \widehat{Y}_{d,e} := \Big(\frac{e-1}{e+1}\Big)^{d/2} Y_{d,e}$$

for $d, e \ge 2$.

Lemma 10 (Row and column control)**.** *Let Q be a coloring of a degree-m slice of f. Then*

$$\sum_{d\ge 2,\, e\ge 1} \frac{\widehat{X}_{d,e}^2}{d^{2B}} \le \Gamma(f; B)^2 \|f\|_\infty^2,$$

and likewise for $\widehat{Y}_{d,e}$.

Proof. Fix $d \ge 2$ and put $\rho_d = \sqrt{(d-1)/(d+1)}$. For each $|A| = d$ let B_A be the full y-row of Q. Minkowski ($q_d \le 2$) and Lemma 6 give

$$\Big(\sum_e \widehat{X}_{d,e}^2\Big)^{1/2} \le \Big[\sum_A \|T_{\rho_d} B_A\|_2^{q_d}\Big]^{1/q_d} \le \Big[\sum_A \|B_A\|_{q_d}^{q_d}\Big]^{1/q_d}.$$

The right-hand side is $\|g_d\|_{q_d} \le \|g_d\|_2$, where $g_d(y) = \|a\big((Q(\,\cdot\,, y))_{=d}\big)\|_{q_d}$. Integrating $\sum_d g_d(y)^2/d^{2B}$ and applying (2) to each further slice $Q(\,\cdot\,, y) \in \mathcal{S}_m$ yields the claim. No bidegree was isolated in L^∞. □

Lemma 11 (Mixed-angle summation [1, Lemma 5.2])**.** *Let $0 < a < 1/2$. If $u_i, v_i \ge 0$ satisfy $\sum u_i \le 1$, $\sum v_i \le 1$, and $\theta_i \in [a, 1-a]$, then*

$$\sum_i u_i^{\theta_i} v_i^{1-\theta_i} \le 2\exp\{-h(a)\}, \qquad h(a) := -a\log a - (1-a)\log(1-a).$$

Proof. Split according to $u_i \ge v_i$ or not. Hölder produces $U_+^{1-a}V_+^{a} + U_-^{a}V_-^{1-a}$. Enlarging unused mass we may assume $U_+ + U_- = V_+ + V_- = 1$. Writing $U_+ = x$, $V_+ = y$ with $0 \le y \le x \le 1$, a second Hölder gives

$$x^{1-a}y^a + (1-x)^a(1-y)^{1-a} \le (1+x-y)^{1-a}(1-x+y)^a.$$

With $\delta = x - y$ the right-hand side is $(1+\delta)^{1-a}(1-\delta)^a$, whose maximum on $[0,1]$ occurs at $\delta = 1-2a$ and equals $2a^a(1-a)^{1-a} = 2e^{-h(a)}$. □

Lemma 12 (Central entropy contraction)**.** *Let $0 < a < 1/2$, $B > 0$, $R \geq 2/a$, and let Q be a coloring of a degree-m slice of f, $m \geq R$. Set $\kappa_R := \sup_{r \geq R}(r+1)^{1/(2r)} = (R+1)^{1/(2R)}$. Then*

$$\Big[\sum_{r=R}^{m} r^{-2B}\Big(\sum_{(d,e)\in\mathcal{C}_r(a)} \|a(Q_{d,e})\|_{q_r}^{q_r}\Big)^{2/q_r}\Big]^{1/2} \leq \kappa_R \cdot 3\sqrt{2} \cdot \exp\big\{-(B+\tfrac{1}{2})h(a)\big\}\,\Gamma(f;B)\,\|f\|_\infty.$$

Proof. Fix $(d,e) \in \mathcal{C}_r(a)$, so $d, e \geq 2$ and $\vartheta = d/r \in [a, 1-a]$. Lemma 4 gives $\|a(Q_{d,e})\|_{q_r} \leq X_{d,e}^{\vartheta} Y_{d,e}^{1-\vartheta}$. The algebra of hats and the two-block computation of Lemma 7 produce

$$X^{\vartheta}Y^{1-\vartheta} \leq 3\,\widehat{X}^{\vartheta}\widehat{Y}^{1-\vartheta}.$$

The weight identity $d^{B\vartheta}e^{B(1-\vartheta)}/r^B = \exp\{-Bh(\vartheta)\} \leq \exp\{-Bh(a)\}$ yields, with $U = \widehat{X}/d^B$ and $V = \widehat{Y}/e^B$,

$$\big(\|a\|_{q_r}/r^B\big)^2 \leq 9\exp\{-2Bh(a)\}\,U^{2\vartheta}V^{2(1-\vartheta)}.$$

Lemma 10 gives $\sum U^2 \leq G^2$ and $\sum V^2 \leq G^2$ with $G = \Gamma(f;B)\|f\|_\infty$. Lemma 11 applied to U^2/G^2 and V^2/G^2 produces

$$\sum_r \sum_{\mathcal{C}_r(a)} \big(\|a\|_{q_r}/r^B\big)^2 \leq 18\exp\{-(2B+1)h(a)\}\,G^2.$$

Since $1/q_r - 1/2 = 1/(2r)$ and $\#\mathcal{C}_r(a) \leq r+1$,

$$\Big(\sum_{\mathcal{C}} \|a\|_{q_r}^{q_r}\Big)^{1/q_r} \leq (r+1)^{1/(2r)}\Big(\sum_{\mathcal{C}} \|a\|_{q_r}^2\Big)^{1/2}.$$

Square, sum in r, and pull out κ_R. Then $\sqrt{18} = 3\sqrt{2}$ and $\frac{1}{2}(2B+1) = B + 1/2$. □

Lemma 13 (Diffuse tail)**.** *Let $0 < a < 1/2$, $B > 0$, and $R \geq 2/a$. Let $p := \inf_{r \geq R} p_r(a) > 0$, and assume*

$$c_R := \kappa_R\, p^{-1/q_R} \cdot 3\sqrt{2} \cdot \exp\big\{-(B+\tfrac{1}{2})h(a)\big\} < 1. \tag{3}$$

Then every degree-m slice g of f, $m \geq R$, satisfies

$$\Big[\sum_{r=R}^{m} \frac{\|a(g^{=r})\|_{q_r}^2}{r^{2B}}\Big]^{1/2} \leq c_R\,\Gamma(f;B)\,\|f\|_\infty.$$

Proof. Color the free coordinates of g once. Write $C_r(\omega)$ for the q_r-norm of the central-window children. Exact ancestry and Lemma 8 give $p\,\|a(g^{=r})\|_{q_r}^{q_r} \leq \mathbb{E}\,C_r(\omega)^{q_r}$. Since $q_r \leq 2$,

$$\|a(g^{=r})\|_{q_r} \leq p^{-1/q_r}\big(\mathbb{E}\,C_r(\omega)^2\big)^{1/2}.$$

Multiply by r^{-B}, square, and sum $r \geq R$. Use $q_r \geq q_R$ so $p^{-1/q_r} \leq p^{-1/q_R}$. Apply Lemma 12 to each colored polynomial, using $\|Q\|_\infty \leq \|f\|_\infty$. □

6. The head, by Markov on slices

Lemma 14 (Head of a slice). *Let $g(x) = f(x, a)$ be a degree-m slice. For $1 \le r < R$ and R fixed,*

$$\|g^{=r}\|_\infty \le \Lambda_{m,r}\|f\|_\infty, \qquad \Lambda_{m,r} := \frac{T_m^{(r)}(1)}{r!} = \frac{m^2(m^2-1)\cdots\big(m^2-(r-1)^2\big)}{r!\,(2r-1)!!}.$$

In particular $\Lambda_{m,r} \le m^{2r}/(r!\,(2r-1)!!)$. Hence

$$\|a(g^{=r})\|_{q_r} \le C_r\Lambda_{m,r}\|f\|_\infty.$$

Proof. Fix a free vertex x. The univariate polynomial $p_x(t) := f(tx, a)$ lies in Π_m, and Lemma 5 gives $\|p_x\|_{L^\infty[-1,1]} \le \|f\|_\infty$. Expanding in Walsh layers,

$$\chi_S(tx, a) = t^{|S\cap\text{free}|}\chi_{S\cap\text{free}}(x)\,\chi_{S\cap\text{frozen}}(a),$$

so $g^{=r}(x) = [t^r]p_x = p_x^{(r)}(0)/r!$. Rotate p_x by a unimodular scalar so that $p_x^{(r)}(0)$ is real and nonnegative, and let $u = \operatorname{Re} p_x$. Then $u \in \Pi_m(\mathbb{R})$, $\|u\|_\infty \le \|p_x\|_\infty$, and $u^{(r)}(0) = p_x^{(r)}(0)$. The Markov brothers' inequality states $\|u^{(r)}\|_{L^\infty[-1,1]} \le T_m^{(r)}(1)\|u\|_\infty$, and $T_m^{(r)}(1)$ is the displayed product. Thus $|g^{=r}(x)| \le \Lambda_{m,r}\|f\|_\infty$. The bound is pointwise on vertices, so it is an L^∞ bound on $g^{=r}$. Apply the homogeneous constant C_r to this layer. For $r < 12$ the number C_r is a fixed DMP constant; there is no n in the estimate.

This is coefficient extraction at 0 for a bounded order r, not the operator $\|P_r\|_{\infty\to\infty}$ and not extraction of a central degree $k \sim m/2$ (the latter costs $\exp(\Theta(m))$ by the Chebyshev coefficient of T_m or the L^1 mass of the degree-k Dirichlet kernel on $\{\pm1\}^{2k}$). □

Remark 15. For an arbitrary inhomogeneous F, the quantity $\|a(F^{=r})\|_{q_r}/\|F\|_\infty$ leaks n. Majority has $W_r = \Theta_r(1)$ spread over $\binom{n}{r}$ coefficients when r is odd, hence $\|a(\mathrm{Maj}^{=r})\|_{q_r} \gtrsim \sqrt{n}$. The paper's bound $\|F_r\|_\infty \le \|F\|_\infty$ is the torus average [1, (4.5)] and is false on the cube. Majority is not a degree-m slice, and is not used.

7. Absorption

Lemma 16 (Contraction at $a = 1/6$, $B = 5$, $R = 12$). *With $a = 1/6$, $B = 5$, $R = 12$ one has $c_{12} < 1/2$.*

Proof. Here $\kappa_{12} = 13^{1/24}$, $p \ge 1 - 2e^{-8/3}$, $q_{12} = 24/13$, and $h(1/6) = \frac{1}{6}\log 6 + \frac{5}{6}\log\frac{6}{5}$. The factors satisfy

$$13^{1/24} < \frac{28}{25}, \qquad 1 - 2e^{-8/3} > \frac{17}{20}, \qquad \Big(\frac{20}{17}\Big)^{13/24} < \frac{11}{10},$$

$$3\sqrt{2} < \frac{17}{4}, \qquad \exp\big(-\tfrac{11}{2}h(\tfrac{1}{6})\big) < \frac{9}{100}.$$

The first is $(28/25)^{24} > 15 > 13$. The second is $e^{8/3} > 40/3$, equivalently $2e^{-8/3} < 3/20$. The third is $(20/17)^{13} < (11/10)^{24}$. The fourth is $(17/4)^2 = 289/16 > 18$. For the fifth, $\log 6 > 7/4$ and $\log(6/5) > 2/11$, so

$$h(\tfrac{1}{6}) > \frac{1}{6}\cdot\frac{7}{4} + \frac{5}{6}\cdot\frac{2}{11} = \frac{7}{24} + \frac{5}{33} = \frac{77+40}{264} = \frac{117}{264} = \frac{39}{88}.$$

Thus $\frac{11}{2}h(1/6) > 39/16 = 2.4375 > \log(100/9)$, and the exponential is $< 9/100$. Therefore

$$c_{12} < \frac{28}{25}\cdot\frac{11}{10}\cdot\frac{17}{4}\cdot\frac{9}{100} = \frac{47124}{100000} < \frac{1}{2}.$$

□

Proof of Theorem 1. Degrees $m < 12$ are a fixed finite list of DMP constants. Let $m \geq 12$ and let f be Walsh-homogeneous of degree m. Lemma 14 with $R = 12$ gives, for every slice g of f,

$$A(f) := \Big[\sum_{r=1}^{11} \frac{\|a(g^{=r})\|_{q_r}^2}{r^{10}}\Big]^{1/2} \leq K_{12}\, m^{22}\|f\|_\infty,$$

where

$$K_{12} = \Big(\sum_{r=1}^{11} \frac{C_r^2}{r^{10}\,(r!\,(2r-1)!!)^2}\Big)^{1/2} < \infty.$$

Lemma 13 and Lemma 16 give a tail at most $\frac{1}{2}\Gamma(f;5)\|f\|_\infty$. Adding the two pieces in (2) produces $\Gamma(f;5) \leq K_{12}m^{22} + \frac{1}{2}\Gamma(f;5)$, hence $\Gamma(f;5) \leq 2K_{12}\, m^{22}$. The original f is a slice with nothing frozen and is purely of degree m, so the left-hand side of (2) is the single term $\|a(f)\|_{q_m}/m^5$. Thus

$$C_m \leq m^5\Gamma(f;5) \leq 2K_{12}\, m^{27}.$$

The constants K_{12} and $1/2$ are independent of n and of f. □

Remark 17 (What is not used). Display [1, (4.5)] (torus projection), the substitution $z_j \mapsto (x_j + y_j)/2$, dominant-power compression [1, Lemma 5.5], and any L^∞ extraction of a single central bidegree. The two-block Lemma 7 is applied only to already bihomogeneous arrays, never to a projected piece with an L^∞ bound inherited from a parent.

8. Inhomogeneous polynomials

Boolean homogenization fails: if $x_0 \in \{\pm 1\}$, then $x_0^2 = 1$, so the substitution $F_r(x) \mapsto x_0^{m-r}F_r(x)$ collapses to a parity and does not produce a degree-m character. The extension uses the same bootstrap on the class of all degree-at-most-m functions.

Definition 18. For each n let $\Gamma_{m,n}(B)$ be the least constant such that every $g : \{\pm 1\}^n \to \mathbb{C}$ of Walsh degree at most m satisfies

$$\Big[\sum_{r=1}^{m} \frac{\|a(g^{=r})\|_{q_r}^2}{r^{2B}}\Big]^{1/2} \leq \Gamma_{m,n}(B)\,\|g\|_\infty.$$

Set $\Gamma_m(B) := \sup_n \Gamma_{m,n}(B)$. This supremum is finite: Lemma 14 applied to $g(tx)$ (equivalently Lemma 19, for every $r \le m$) gives $\|g^{=r}\|_\infty \le \Lambda_{m,r}\|g\|_\infty$, so

$$\Gamma_{m,n}(B) \le \Big(\sum_{r=1}^{m} \frac{(C_r\Lambda_{m,r})^2}{r^{2B}}\Big)^{1/2} \le K_m < \infty$$

uniformly in n, with K_m of size $m^{O(m)}$. The bootstrap below replaces K_m by a polynomial.

Lemma 19 (Inhomogeneous head)**.** *Let g have Walsh degree at most m. For $1 \le r < R$,*

$$\|g^{=r}\|_\infty \le \Lambda_{m,r}\|g\|_\infty,$$

with the same $\Lambda_{m,r}$ as in Lemma 14. Hence $\|a(g^{=r})\|_{q_r} \le C_r\Lambda_{m,r}\|g\|_\infty$.

Proof. A Walsh polynomial is multilinear. For a vertex x, the univariate $p_x(t) := g(tx) = \sum_{k=0}^{m} t^k g^{=k}(x)$ lies in Π_m, and Lemma 5 gives $\|p_x\|_{L^\infty[-1,1]} \le \|g\|_\infty$. The rest of the proof of Lemma 14 applies verbatim, with g in place of the slice of a homogeneous parent. No extra variable is introduced. □

Lemma 20 (The class is closed)**.** *Fair 2-coloring and freezing coordinates send degree-at-most-m functions to degree-at-most-m functions. Consequently Lemmas 10, 12, and 13 hold with $\Gamma(f;B)$ replaced by $\Gamma_m(B)$ and $\|f\|_\infty$ replaced by $\|g\|_\infty$, for an arbitrary degree-at-most-m test function g. Capture (Lemma 8) still runs separately on each Walsh level.*

Proof. Coloring is a relabeling of coordinates. Freezing a coordinate at ± 1 does not raise degree. The proofs of the three lemmas never used homogeneity of the parent except to identify the right-hand side with $\|f\|_\infty$; that is now $\|g\|_\infty$ by definition of $\Gamma_m(B)$. Exact ancestry is a statement about each multi-index S separately, so it applies to each level $g^{=r}$. □

Proof of Theorem 2. Take $B = 5$, $a = 1/6$, $R = 12$. Lemma 19 gives, uniformly in n and in g,

$$\Big[\sum_{r=1}^{11} \frac{\|a(g^{=r})\|_{q_r}^2}{r^{10}}\Big]^{1/2} \le K_{12}\, m^{22}\|g\|_\infty,$$

with the same K_{12} as in the proof of Theorem 1. Lemma 20 and Lemma 16 give a tail at most $\frac{1}{2}\Gamma_m(5)\|g\|_\infty$. Therefore

$$\Gamma_{m,n}(5) \le K_{12}m^{22} + \tfrac{1}{2}\Gamma_m(5)$$

for every n. Taking $\sup_n$ and using $\Gamma_m(5) < \infty$ on each finite cube (then passing to the supremum) yields $\Gamma_m(5) \le 2K_{12}m^{22}$. This is (1). In particular

$$\|a(F^{=r})\|_{q_r} \le 2K_{12}\, m^{22}\, r^5\|F\|_\infty \le Km^{27}\|F\|_\infty \qquad (1 \le r \le m).$$

Since $q_m \geq q_r$, the comparison $\|a(F^{=r})\|_{q_m} \leq \|a(F^{=r})\|_{q_r}$ holds, and

$$\sum_{|S|\leq m} |\widehat{F}(S)|^{q_m} = \sum_{r=0}^{m} \|a(F^{=r})\|_{q_m}^{q_m} \leq |\widehat{F}(\emptyset)|^{q_m} + \sum_{r=1}^{m} \|a(F^{=r})\|_{q_r}^{q_m}.$$

The constant term is at most $\|F\|_\infty^{q_m}$, and each of the m remaining layers is at most $(Km^{27}\|F\|_\infty)^{q_m}$. Thus

$$\widetilde{C}_m \leq \left(1 + m(Km^{27})^{q_m}\right)^{1/q_m} \leq K' m^{27+1/q_m} \leq K' m^{28},$$

because $1/q_m = (m+1)/(2m) \leq 3/4$ for $m \geq 2$. The case $m = 1$ is $C_1 = \widetilde{C}_1 = 1$. $\square$

References

[1] D. M. Pellegrino and E. V. Teixeira, *Polynomial growth of complex polynomial Bohnenblust–Hille constants*, arXiv:2608.16584, 2026.